Indian Statistical Institute
B. Math. Hons. III Year
Semestral Examination 2002-2003
Analysis IV

Date:28-04-2003 Instructor: A. Sitaram

1. Let X be a compact subset of $\boldsymbol{R}^2$. Let $f_{n,m}(x,y) = e^{nx+my}$, where n, m are non-negative integers, and let $g_{n,m}$ be the restriction of $f_{n,m}$ to X. Let A be the set of finite linear combinations of functions of the type $g_{n,m}$. Prove that A is dense in $C(X)$. [15]

2. X and Y are metric spaces. Let F be a family of equicontinuous real valued functions on Y and h a continuous function from X into Y. Prove that $G = \{f \circ h | f \in F\}$ is an equicontinuous family on X. [10]

3. Prove that given $\epsilon > 0$, there exists a dense open subset O_ϵ of $\boldsymbol{R}$ such that the Lebesgue measure of O_ϵ is less than ϵ. [10]

4. f is a measurable function on $[0,1]$, $f(x) > C$ almost everywhere on $[0,1]$, where C is a positive constant. Prove that $\int_{[0,1]} f > C$. [10]

5. Define f on the open interval $(0,1)$ as follows:
$f(x) = 0$ if x is irrational
$f(x) = \frac{1}{q}$ if $x = \frac{p}{q}$, where p and q are integers with no common factors.
Prove that f is measurable. [10]

6. Let H be a separable Hilbert space. If $\{x_n\}$ is a sequence in H such that $(x_n, x) \to 0$ for every $x \in H$, does it necessarily follow that $\|x_n\| \to 0$? Prove, if true. Give a counter example if false. (In the above, (,) denotes inner product.) [15]